Quantum Pseudo-Telepathy and the Kochen-Specker Theorem

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Abstract — **There are different approaches to proving the impossibility of classical hidden-variable explanations of quantum-mechanical behavior. Whereas Kochen and Specker proved that a three- or higher-dimensional quantum-mechanical system cannot be "classically" prepared for all possible alternative measurements in a consistent way, Bell showed that the behavior of certain two-partite systems is non-local, i.e., inexplicable by shared classical information. We show a close connection between deterministic manifestations of such non-locality—called "pseudo-telepathy" games—and Kochen and Specker's theorem: Every such game leads to a Kochen-Specker contradiction, and *vice versa*.**

I. Contextuality, Non-Locality, ...

The state of a quantum-mechanical system is a unit vector $|\Psi\rangle$ in some Hilbert space $\mathcal{H}$, and a possible (von Neumann) measurement on the state is defined by an orthonormal basis $b = \{|u_1\rangle, |u_2\rangle, \ldots\}$ of $\mathcal{H}$. The measurement outcome is then one of the unit vectors in b (i.e., its number)—this is the state in which the system will be after the measurement—, where a certain measurement result i has non-zero probability if and only if $\langle\Psi|u_i\rangle \neq 0$ holds.

Initiated by [3], it has been a central objective in the history of quantum mechanics to embed quantum theory into a classical theory based on so-called *hidden variables* which, roughly speaking, describe a system's "behavior" under all possible alternative measurements that can be carried out on it. Specker [5] and later Kochen and Specker [4] showed that this is impossible or, more precisely, that any such description has to be "contextual": It is impossible to assign values 0 and 1 to all unit vectors in the three-dimensional Hilbert space $\mathcal{H} = \mathbf{C}^3$ in such a way that every orthonormal basis contains *exactly one* vector with value 1—this vector would be the corresponding measurement outcome. Interestingly, Kochen and Specker showed that this impossibility can already hold with respect to a finite set of vectors. We call a *KS set* a set of vectors such that every prediction function f assigns 1 to two orthogonal vectors.

A different approach to showing the impossibility of hidden-variable explanations for the behavior of quantum systems was taken by Bell [1]. According to quantum mechanics, two two-dimensional systems, called *quantum bits*, can—even when physically separated—be in a joint state which cannot be completely described by giving the states of the two bits separately; such a state is called *entangled*. An example was given by Einstein, Podolsky, and Rosen [3] as $|\Phi^+\rangle := (|00\rangle + |11\rangle)/\sqrt{2}$. Bell showed that the joint behavior with respect to different measurements on the two subsystems of this state cannot be explained by shared classical information under the assumption that no communication is allowed between the two parts of the system. More precisely, Bell derived certain inequalities—the *Bell inequalities*—that are satisfied for all systems the behavior of which *do* have a classical explanation; he then showed that they are violated by the behavior of the EPR state $|\Phi^+\rangle$. This *non-locality* or "Spukhafte Fernwirkung—spooky action at a distance—", although it does not allow the parties controlling the distant systems for (instant) message transmission, implies that no classical hidden-variable theory can explain their behavior.

"Pseudo-telepathy" [2] is a deterministic version of non-local behavior. More precisely, a *PT game* is a bipartite quantum state $|\Psi\rangle \in \mathcal{H} \otimes \mathcal{H}$ together with a set B of orthonormal bases such that the following holds: For every pair of functions $f_1, f_2 : B \to \bigcup B$ with $f_i(b) \in b$ for all $b \in B$—this is a local classical strategy—there exists a specific pair $(b_1, b_2) \in B^2$ such that the outcome pair $(f_1(b_1), f_2(b_2))$ has probability 0 if the state $|\Psi\rangle$ is measured with respect to the basis $b_1 \otimes b_2$. In other words, this condition means that however two parties, unable to communicate, try to simulate the behavior of the quantum state—without actually sharing it—, they will finally be caught because they produce an invalid output pair.

II. ... and Their Connection

KS sets and *PT games* are closely related. More precisely, any game with respect to the state $|\Psi_n\rangle := (|00\rangle + |11\rangle + \cdots + |n-1, n-1\rangle)/\sqrt{n}$ leads to a KS set, and *vice versa*.

Theorem 1 *Let B be a set of orthonormal bases of $\mathcal{H} = \mathbf{C}^n$ that is closed under complex conjugation. Then $|\Psi_n\rangle$ together with B is a PT game if and only if the union of the sets of B is a KS set.*

The fact that KS sets exist in $\mathbf{C}^n$ if and only if $n \geq 3$ holds implies, by Theorem 1, that pseudo-telepathy is possible with a shared maximally entangled quantum *trit* pair, but impossible if only a maximally entangled quantum *bit* pair is shared.

Acknowledgments

The authors thank E. Specker for many interesting and inspiring discussions.

References

[1] J. S. Bell, On the Einstein-Podolsky-Rosen paradox, *Physics*, Vol. 1, pp. 195–200, 1964.

[2] G. Brassard, R. Cleve, and A. Tapp, The cost of exactly simulating quantum entanglement with classical communication, *Phys. Rev. Lett.*, Vol. 83, No. 9, pp. 1874–1878, 1999.

[3] A. Einstein, B. Podolsky, and N. Rosen, Can quantum-mechanical description of physical reality be considered complete?, *Phys. Rev.*, Vol. 41, 1935.

[4] S. Kochen and E. Specker, The problem of hidden variables in quantum mechanics, *Journal of Mathematics and Mechanics*, Vol. 17, No. 1, 1967.

[5] E. Specker, Die Logik nicht gleichzeitig entscheidbarer Aussagen, *Dialectica*, Vol. 14, pp. 239–246, 1960.

[1] Supported by SNF No. 20-66716.01.

[2] Supported by Canada's NSERC.